The effect of a magnetic field on the flow of a conducting fluid past a circular disk

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In the previous paper (Chester 1961) it was shown that, for large values of the Hartmann number, the asymptotic solution for the flow past a body of revolution has a discontinuity on the surface of a cylinder which circumscribes the body. The flow in the region of this discontinuity is now investigated in more detail when the body is a circular disk broadside-on to the flow. It will be shown that there is actually a region of transition whose thickness is $O(|x|^{\frac{1}{2}}/M^{\frac{1}{2}})$, where x is the axial distance from the disk and M is the Hartmann number. This region is thin near the disk, but gradually thickens until it merges into the over-all flow field for $x = O(M)$.

The leading terms in the expression for the drag are given by

$$\frac{D}{D_s} = \frac{M\pi}{8}\left(1+\frac{2}{M}\right),$$

where D_s is the Stokes drag.

1. Introduction

We consider here a special case of the problem investigated in the preceding paper (Chester 1961, henceforth this paper is denoted by C). The same approximations are assumed to be valid, and the body is taken to be a circular disk of radius a, with its axis parallel to the uniform streaming motion at infinity. It will be shown that the mathematical problem for this particular case can be discussed in greater detail than the more general arguments of the previous investigation allowed. In particular the properties of the shear layer, which for $M \gg 1$ must exist at the periphery of the circumscribing cylinder, can now be discussed, and this is the main purpose of this paper. Since it has been shown that the flow field for large values of the Hartmann number is insensitive to the detailed shape of the body, it may reasonably be assumed that the results also describe the situation for a more general class of bodies.

2. The basic equations

The non-dimensional velocity and pressure are given by equations (C 14)–(C 17), namely

$$\mathbf{V} = \mathbf{i} + e^{Mx}\nabla\phi_1 + e^{-Mx}\nabla\phi_2, \tag{1}$$

$$p = M e^{Mx}\frac{\partial\phi_1}{\partial x} - M e^{-Mx}\frac{\partial\phi_2}{\partial x}, \tag{2}$$

where
$$\nabla^2\phi_1 + M\frac{\partial \phi_1}{\partial x} = 0, \tag{3}$$

$$\nabla^2\phi_2 - M\frac{\partial \phi_2}{\partial x} = 0, \tag{4}$$

and the length parameter used in the definition of M is now the radius of the disk. Because of symmetry the radial component of $\mathbf{V}$ must be zero on $x = 0$, and this implies that $\phi_1 + \phi_2 = 0$ on $x = 0$. Thus $(\phi_1 + e^{-Mx}\phi_2)$ satisfies the equation

$$\left(\nabla^2 + M\frac{\partial}{\partial x}\right)(\phi_1 + e^{-Mx}\phi_2) = 0, \tag{5}$$

and is zero on $x = 0$. It is therefore identically zero, and so we write

$$\phi_1 e^{mx} = -\phi_2 e^{-mx} = \phi \tag{6}$$

say, where $2m = M$ and
$$(\nabla^2 - m^2)\,\phi = 0, \tag{7}$$

$$\mathbf{V} = \{1 - 2m\phi\cosh mx\}\,\mathbf{i} + 2\sinh mx\nabla\phi, \tag{8}$$

$$\frac{p}{2m} = 2\frac{\partial \phi}{\partial x}\cosh mx - 2m\phi\sinh mx. \tag{9}$$

The remaining boundary conditions to be applied are that

$$\mathbf{V} = 0 \quad \text{for} \quad x = 0 \quad \{\eta = (y^2+z^2)^{\frac{1}{2}} < 1\}.$$

Also, it is consistent with the equations and boundary conditions to look for a solution such that ϕ is symmetrical, and hence p is anti-symmetrical, about $x = 0$. Since p is also continuous for $\eta > 1$, we must have

$$p = 0 \quad \text{for} \quad x = 0 \quad (\eta > 1).$$

In terms of ϕ, these boundary conditions imply that

$$\left.\begin{aligned} \phi &= \frac{1}{M} \quad \text{for} \quad x = 0 \quad (\eta < 1), \\ \frac{\partial \phi}{\partial x} &= 0 \quad \text{for} \quad x = 0 \quad (\eta > 1). \end{aligned}\right\} \tag{10}$$

3. Approximate solution for large Hartmann number

Any solution of (7), which is continuous together with its derivatives inside a closed surface S, may be written

$$\phi(P) = -\frac{1}{4\pi}\iint_S \left(G\frac{\partial \phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right) dS, \tag{11}$$

where n is the inward-drawn normal to S, P is a point within S, and G is another function satisfying the same conditions as ϕ save that, near $P, G \sim \rho^{-1}$ where ρ is the distance from P. If, in addition $\partial G/\partial n = 0$ on S, then G is called the Green's function for this particular problem and

$$\phi(P) = -\frac{1}{4\pi}\iint_S G\frac{\partial \phi}{\partial n}\,dS. \tag{12}$$

When S is the plane $x = 0$, together with a large hemisphere in the positive half-plane on which the integral tends to zero as the radius tends to infinity, then

$$G = \frac{\exp\left[-m\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{\frac{1}{2}}\right]}{\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{\frac{1}{2}}} + \frac{\exp\left[-m\{(x+x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{\frac{1}{2}}\right]}{\{(x+x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{\frac{1}{2}}}, \tag{13}$$

where P is the point (x_0, y_0, z_0).

Thus, if

$$\frac{\partial\phi}{\partial x} = \chi(y,z) \quad \text{on} \quad x = 0,$$

we have, omitting the suffix,

$$\phi(x,y,z) = -\frac{1}{2\pi}\iint \chi(y',z')\frac{\exp\left[-m\{x^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}\right]}{\{x^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}}\,dy'\,dz'. \tag{14}$$

Consider now the argument used in C. We note first that, by (10), $\chi = 0$ except on the surface of the disk. Hence, for large m, $e^{mx}\,\phi$ is exponentially small unless a line through (x, y, z) parallel to the x-axis meets the surface of the disk. If it does, then the sensible contribution to the integral comes from the neighbourhood of the point $(0, y, z)$. Thus we write, approximately

$$\phi(x,y,z) = -\frac{\chi(y,z)}{2\pi}\iint\frac{\exp\left[-m\{x^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}\right]}{\{x^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}}\,dy'\,dz' \tag{15}$$

for points inside the cylinder circumscribing the disk. The argument then proceeds by integrating over the whole plane $-\infty < y' < \infty$, $-\infty < z' < \infty$. This gives

$$\phi(x,y,z) = -\frac{1}{m}e^{-mx}\chi(y,z), \tag{16}$$

and since, by (10),

$$\phi = \frac{1}{2m} \quad \text{for} \quad x = 0 \quad [(y^2+z^2) < 1],$$

we have, finally, $\chi = -\frac{1}{2}$ and so

$$\phi = \frac{1}{2m}e^{-mx} \quad \text{for} \quad x > 0.$$

In particular, by (9), $p = -2m$, which agrees with the result in C.

The argument fails for points near the surface of the circumscribing cylinder. It makes χ discontinuous at the edge of the disk, and the variation of χ in this region will affect the integral in (14) significantly if (y^2+z^2) is close to unity. However, when m is large, any point for which this is true must lie within a distance $O(m^{-1})$ from the surface $y^2+z^2 = 1$, otherwise the exponential factor in the integrand of (14) will be small enough to make the error in χ unimportant. (This is true only for $x \ll M$. Further comment on this point will appear later.) On such a length scale, the radius of curvature of the edge of the disk is large, and the previous argument can be refined by replacing the disk by a *semi*-infinite plane. The edge of this plane is to be tangential to the disk at the point of its edge nearest the point (x, y, z) at which ϕ is being evaluated. It is sufficient to consider only points $(x, y, 0)$; the value of ϕ at other points is then deducible from symmetry. The disk is then replaced by the semi-infinite plane $x = 0, y < 1$.

To return to the original equations, the problem now presents itself in the following form. A two-dimensional solution of $(\nabla^2 - m^2)\phi = 0$ is required under the boundary conditions

$$\left.\begin{aligned} \phi &= \frac{1}{M} \quad \text{for} \quad x = 0 \quad (-\infty < y < 1), \\ \frac{\partial \phi}{\partial x} &= 0 \quad \text{for} \quad x = 0 \quad (1 < y < \infty). \end{aligned}\right\} \tag{17}$$

We also require that $\phi \to 0$ at infinity, except for $x = 0$, $y \to -\infty$. This last qualification is necessary because of the artificial boundary condition now imposed on the plane $x = 0$. This will not incur any serious error since we are interested in the solution only in the vicinity of $y = 1$.

Problems similar to this have been familiar in diffraction theory for some time (Gunn 1947; Chester 1950), and it is not difficult to derive the appropriate solution in the present case.

We first substitute $x = r\cos\theta$, $(y-1) = r\sin\theta$, so that the two sides of the semi-infinite plane become $\theta = -\frac{1}{2}\pi$, $\theta = \frac{3}{2}\pi$. Then the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = m^2 \phi$$

has solutions of the form

$$e^{\pm mx} \int^{(2r)^{\frac{1}{2}} \sin \frac{1}{2}(\theta - \theta_0)} e^{-m\lambda^2} d\lambda. \tag{18}$$

These solutions are most easily obtained by transforming to parabolic coordinates (Lamb 1932; Gunn 1947). If the various possible solutions are combined, it is found that the appropriate function satisfying all the boundary conditions is

$$\phi = \frac{1}{2}\left(\frac{1}{m\pi}\right)^{\frac{1}{2}} \left\{ e^{-mx} \int_{(2r)^{\frac{1}{2}} \sin \frac{1}{2}\theta}^{\infty} e^{-m\lambda^2} d\lambda + e^{mx} \int_{(2r)^{\frac{1}{2}} \cos \frac{1}{2}\theta}^{\infty} e^{-m\lambda^2} d\lambda \right\} \quad (-\tfrac{1}{2}\pi < \theta < \tfrac{3}{2}\pi). \tag{19}$$

This is the unique solution of the problem as presented. It is continuous in the whole plane, but $\partial\phi/\partial x$ is discontinuous across $x = 0$, $y < 1$. On the plane $x = 0$, ϕ is essentially zero for $y > 1$ and $1/2m$ for $y < 1$ with an error which is exponentially small except near $y = 1$. There $\phi = 1/2m$ and is continuous in its neighbourhood. Also $\partial\phi/\partial x = 0$ for $x = 0$, $y > 1$, and for $y < 1$, $\theta = -\frac{1}{2}\pi$,

$$\frac{p}{2M} = \frac{\partial \phi}{\partial x} = -\frac{1}{2} + \frac{1}{\pi^{\frac{1}{2}}} \int_{m^{\frac{1}{2}}(1-y)^{\frac{1}{2}}}^{\infty} e^{-\lambda^2} d\lambda - \frac{e^{-m}(1-y)}{2\{m\pi(1-y)\}^{\frac{1}{2}}}. \tag{20}$$

Thus $\partial\phi/\partial x = -\frac{1}{2}$ with an error which is exponentially small except near $y = 1$ where $\partial\phi/\partial x$ has an integrable singularity. It is also discontinuous across $x = 0$, $y < 1$, taking equal and opposite values at corresponding points on the two sides.

Away from the disk, say for $x > 1$, equations (8) and (9) show that the velocity and pressure are given asymptotically by

$$\left.\begin{aligned} \mathbf{V}-\mathbf{i} &\sim e^{2mx}\nabla(e^{-mx}\phi), \\ p/2m &\sim e^{2mx}\frac{\partial}{\partial x}(e^{-mx}\phi). \end{aligned}\right\} \tag{21}$$

Now, from (19)

$$e^{2mx}\frac{\partial}{\partial x}(e^{-mx}\phi) \sim -\left(\frac{1}{\pi}\right)^{\frac{1}{2}}\int_{(2mr)^{\frac{1}{2}}\sin\frac{1}{2}\theta}^{\infty} e^{-\lambda^2}\,d\lambda, \tag{22}$$

$$e^{mx}\frac{\partial\phi}{\partial y} = -\frac{1}{2}\left(\frac{1}{2m\pi r}\right)^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta)\,e^{-2mr\sin^2\frac{1}{2}\theta}, \tag{23}$$

which shows that there is a region of sharp transition in the neighbourhood of $\theta = 0$, across which p/M and the x-component of $(\mathbf{V}-\mathbf{i})$ change rapidly from -1 $(\theta < 0)$ to 0 $(\theta > 0)$. The y-component of $\mathbf{V}$ is exponentially small everywhere save in the region of transition. This region contains the points for which $mr\sin^2\frac{1}{2}\theta$ is not large, and so is roughly parabolic in profile with thickness $O(|x|^{\frac{1}{2}}/M^{\frac{1}{2}})$ (by symmetry there is a similar region on the negative side of the plane $x = 0$). Thus it is thin where $|x| = O(M)$ but gradually thickens until it is smeared out completely for $|x| = O(M)$.

When the solution is applied to the circular disk, the co-ordinate r is to be interpreted as the distance from the field point considered to the nearest point on the edge of the disk, the angle θ is the inclination of this line to the x-axis. Some comment is also necessary on the validity of the solution when applied to the disk problem. It is correct only in the shear layer, and only where that layer is thin, so that the effects of curvature are of secondary importance in determining the nature of the flow. It is not really correct to regard it as extending beyond this region, although for $x \ll M$, it does agree with what the results of C would lead one to expect.

The situation resembles that in boundary-layer theory. The equations of that theory are assumed valid in a thin region of the flow, and outside this region it is assumed that there is a known main stream flow. The latter is used as a boundary condition for the edge of the boundary layer, and it is a common procedure to apply this boundary condition at infinity. In the present problem the 'main stream flow' can be thought of as that given by the more elementary theory of C, both inside and outside the circumscribing cylinder. This provides the boundary conditions at infinity for the problem of the semi-infinite plane.

The breakdown of the solution for large values of x also has its analogue in boundary-layer theory and can be explained by the fact that the radial scale of length in the transition layer eventually becomes comparable with the length scale in the other directions. Alternatively by reference to (14) it will be seen that as $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ increases, the sensible contribution to the integral depends more and more on the over-all value of χ, rather than its value in some small neighbourhood of the disk. In fact, for $r \gg M$ it is readily seen from (14) that

$$\phi \sim -\frac{e^{-mr}}{2\pi r}\iint\chi(y',z')\,dy'\,dz' \sim \frac{e^{-mr}}{4r},$$

where the integration is now taken over the plane of the disk. If this value of ϕ is substituted in (8), it follows that the perturbation velocity components tend to zero exponentially at infinity in every direction except the x-direction, where

$$|\mathbf{V}-\mathbf{i}| = O\left\{\frac{m\,e^{-m(r-x)}}{r}\right\}.$$

Finally we may add that a uniformly valid solution for ϕ could be obtained from (14), since the value of $\chi(y', z')$ on the disk is now known. The results will not be given since all the essential information has already been obtained from the 'boundary-layer' theory. The qualitative picture is essentially that given above. There is a thin shear layer in the neighbourhood of the circumscribing cylinder, which gradually thickens as $|x|$ increases until eventually it fuses into the main stream flow for $|x| = O(M)$.

The drag for large values of M is easily obtained by integrating the pressure (given by (20), with y now interpreted as the radial co-ordinate) over the surface of the disk. Note that the pressure over the interior of the disk contributes a term of more significant order to the integrated pressure than the contribution from the edge effect. Now the pressure over the interior is in error by an exponentially small amount. Near the edge the error is more significant, but of less significance than the terms already displayed in (20). This means that (20) should yield the first two leading terms in the drag. When evaluated, they give

$$\frac{D}{\rho\nu aU} = 2M\pi(1+2/M). \tag{24}$$

4. The validity of the basic equations

In C the equation of motion is derived in the form (C 9)

$$R(\mathbf{V}.\nabla)\,\mathbf{V} = -\nabla p + \nabla^2\mathbf{V} + M^2(\mathbf{V}\wedge\mathbf{i})\wedge\mathbf{i}, \tag{25}$$

and the whole of the analysis is based on the assumption that the term on the left-hand side of this equation may be neglected. When $M \ll 1$, this is valid provided $R \ll M$. For the term $M^2(\mathbf{V}\wedge\mathbf{i})\wedge\mathbf{i}$ modifies the Stokes solution by a factor $\{1+O(M)\}$ (Chester, 1957) whereas the term $R(\mathbf{V}.\nabla)\,\mathbf{V}$ modifies the solution by a factor $\{1+O(R)\}$.

We consider here the validity of the assumption for $M \gg 1$, and the significant region is then the shear layer. From (19) one can deduce that in this layer,

$$\begin{aligned}
&\text{the axial component of velocity} &&= O(1),\\
&\text{the radial component of velocity} &&= O(M^{-\frac{1}{2}}),\\
&\text{the pressure} &&= O(M),\\
&\partial/\partial x &&= O(1),\\
&\partial/\partial \eta &&= O(M^{\frac{1}{2}}).
\end{aligned}$$

It then follows that the four terms in (25) have respectively the following orders of magnitude for the axial component of that equation,

$$R,\quad M,\quad M,\quad 0$$

and, for the radial component,

$$RM^{-\frac{1}{2}}, \quad M^{\frac{3}{2}}, \quad M, \quad M^{\frac{3}{2}}.$$

Thus the same condition, namely $R \ll M$, is also required for $M \gg 1$. It is worth noting that, for $M \gg R$ and $M \gg 1$, equation (25) simplifies to

$$\frac{\partial p}{\partial x} = \frac{\partial^2 V_x}{\partial \eta^2}, \quad \frac{\partial p}{\partial \eta} = -M^2 V_\eta,$$

where $\mathbf{V} = (1+V_x, V_\eta)$. One may verify that the approximate solutions do, in fact, satisfy these equations. Note also that the second of these equations explains the mechanism by which the large pressure gradient across the shear layer is maintained by balancing the Lorentz force.

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